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THE DEGREE THEOREM IN HIGHER RANK

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Abstract

Let N be any closed, Riemannian manifold. In this paper we prove that, for most locally symmetric, nonpositively curved Riemannian manifolds M, and for every continuous map $f : N \to M$, the map f is homotopic to a smooth map with Jacobian bounded by a universal constant, depending (as it must) only on Ricci curvature bounds of N. From this we deduce an extension of Gromov's Volume Comparison Theorem for negatively curved manifolds to (most) nonpositively curved, locally symmetric manifolds.

1. Introduction

The problem of relating volume to degree for maps between Riemannian manifolds is a fundamental one. Gromov's Volume Comparison Theorem [14] gives such a relation for maps into negatively curved manifolds. In this paper we extend Gromov's theorem to locally symmetric manifolds of nonpositive curvature. We derive this as a consequence of the following result, which we believe to be of independent interest.

Theorem 1.1 (Universal Jacobian bound)**.** *Let M be a closed, locally symmetric n-manifold with nonpositive sectional curvatures. Assume that M has no local direct factors locally isometric to $\mathbf{R}, \mathbf{H}^2$, or $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. Then for any closed Riemannian manifold N and any continuous map $f : N \to M$, there exists a constant $C > 0$, depending only on n and the smallest Ricci curvatures of N and M, so that f is homotopic to a C^1 map $F : N \to M$ satisfying*

$$|\operatorname{Jac} F| \le C.$$

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Remark. By scaling the metrics it is easy to see that the dependence of the constant C on the smallest curvatures cannot be removed. Actually, we determine the constant explicitly in terms of the symmetric space and the volume entropy of N (see §2.1).

Theorem 1.1 together with a simple degree argument (see §6) gives the following generalization of Gromov's Volume Comparison Theorem.

Theorem 1.2 (The Degree Theorem)**.** *Let M be as in Theorem 1.1. Then for any closed Riemannian manifold N and any continuous map $f : N \to M$,*

$$\deg(f) \leq C \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)}$$

where $C > 0$ depends only on n and the smallest Ricci curvatures of N and M.

Remarks.

1. As tori have self-maps of arbitrary degree, it is easy to see that the theorem would be false without the "no $\mathbf{R}$ factors" hypothesis. We believe that the "no $\mathbf{H}^2$ or $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$ local factors" hypothesis is unnecessary; we show in Example 4.6 below and in §6 of [10], however, that the issue is rather delicate.

2. As with Theorem 1.1, the dependence of the constant C on the smallest curvatures cannot be removed; this dependence is determined explicitly in §2.1.

3. In §6.2 we extend Theorem 1.2 to the case where N and M have finite volume (with "bounded geometry") but are not necessarily compact, and where f is a coarse Lipshitz map.

When $\dim(M) = 2$ the conclusion of the theorem follows easily from the Gauss-Bonnet Theorem. More generally, when M is any closed manifold with positive Gromov norm, Gromov has shown (see [14], p. 8 and the Main Inequality on p. 12) that a degree theorem as in Theorem 1.2 holds for M. Positivity of the Gromov norm for closed, locally symmetric M as in Theorem 1.2 is still an open question. However, this positivity was proved by Savage [21] for closed M locally isometric to $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R})$.

When $\mathrm{rank}(M) = 1$, Besson-Courtois-Gallot [2] proved the stronger *entropy rigidity theorem*, giving the exact best constant C. Entropy rigidity is still open in higher rank [1] ; this would correspond to the above theorem with the constant C in the inequality being $C = \left(\frac{h(g)}{h(g_0)}\right)^n$, where $h(g)$ and $h(g_0)$ are the volume entropies of N and M (see [2]), with equality being obtained iff N is a Riemannian cover of M after an appropriate rescaling.

The Besson-Courtois-Gallot technique is a central ingredient here; indeed the main idea in our proof of Theorem 1.1 is to establish a higher rank version of the "canonical map" of [2], and to give an *a priori* bound on its Jacobian. Obtaining this bound is the hardest part of the present paper (see §4 and §5). Our estimates in §4 and §5 can be viewed as a first step towards proving higher rank entropy rigidity.

The Minvol invariant. One of the basic invariants associated to a smooth manifold M is its *minimal volume*:

$$\mathrm{Minvol}(M) := \inf_g \{\mathrm{Vol}(M, g) : |K(g)| \leq 1\}$$

where g ranges over all smooth metrics on M and $K(g)$ denotes the sectional curvature of g. The basic questions about $\mathrm{Minvol}(M)$ are: for which M is $\mathrm{Minvol}(M) > 0$? When is $\mathrm{Minvol}(M)$ realized by some metric g?

When a nonpositively curved manifold M has a local direct factor locally isometric to $\mathbf{R}$, it is easy to see that $\mathrm{Minvol}(M) = 0$. By taking f to be the identity map (while varying the metric g on M), Theorem 1.2 immediately gives:

Corollary 1.3 (Positivity of Minvol)**.** *Let M be any finite volume, locally symmetric n-manifold ($n \geq 2$) of nonpositive curvature. If M has no local direct factors locally isometric to $\mathbf{R}, \mathbf{H}^2$, or $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$, then $\mathrm{Minvol}(M) > 0$.*

For compact M, Corollary 1.3 was proved (without the $\mathbf{H}^2$ and $\mathrm{SL}(3, \mathbf{R})$ restriction) in [15] (see also [21] for the case manifolds locally modelled on the symmetric space for $\mathrm{SL}(n, \mathbf{R})$). When M admits a (real) hyperbolic metric, Besson-Courtois-Gallot [2] proved that $\mathrm{Minvol}(M)$ is *uniquely* realized by the hyperbolic metric. It seems possible that this might hold in general.

[1] Entropy rigidity has recently been proved [4, 9] for manifolds locally modelled on products of rank one symmetric spaces with no $\mathbf{H}^2$ factors.

Self maps and the co-Hopf property. As $\deg(f^n) = \deg(f)^n$, an immediate corollary of Theorem 1.2 is the following.

Corollary 1.4 (Self maps)**.** *Let M be a finite-volume locally symmetric manifold as in Theorem 1.1. Then M admits no self-maps of degree > 1. In particular, $\pi_1(M)$ is* co-Hopfian: *every injective endomorphism of $\pi_1(M)$ is surjective.*

Note that Corollary 1.4 may also be deduced from Margulis's Superrigidity theorem (for higher rank M). The co-Hopf property for lattices was first proved by Prasad [20].

More generally, if N and M are as in Theorem 1.2 and $f : N \to M$ and $g : M \to N$ are two maps of nonzero degree then $|\deg(f)| = 1 = |\deg(g)|$ since $f \circ g$ is a self map of M.

Outline of the proof of Theorems 1.1 and 1.2. As noted above, a simple degree argument shows that it is enough to prove Theorem 1.1. Given $f : N \to M$ as in the hypothesis of the theorems, we use the method of [2, 3] to construct a "canonical" map $F : N \to M$ which is homotopic to f (hence $\deg F = \deg f$) and has universally bounded Jacobian.

Step 1 (Constructing the map): First consider the case when the metric on N is nonpositively curved. Denote by Y (resp. X) the universal cover of N (resp. M). Let $\mathcal{M}(\partial Y), \mathcal{M}(\partial X)$ denote the spaces of atomless probability measures on the visual boundaries of the universal covers Y, X.

Morally what we do, following the method of [3], is to define a map

$$\widetilde{F} : Y \to \mathcal{M}(\partial Y) \xrightarrow{\phi_*} \mathcal{M}(\partial X) \xrightarrow{\text{bar}} X$$

where $\phi_* = \partial \widetilde{f}_*$ is the pushforward of measures and bar is the "barycenter of a measure" (see §3). The inclusion $Y \to \mathcal{M}(\partial Y)$, denoted $x \mapsto \mu_x$, is given by the construction of the *Patterson-Sullivan measures* $\{\mu_x\}_{x \in X}$ corresponding to $\pi_1(N) < \mathrm{Isom}(Y)$ (see §2). An essential feature of these constructions is that they are all canonical, so that all of the maps are *equivariant*. Hence $\widetilde{F}$ descends to a map $F : N \to M$.

One problem with this construction outline is that the metric on Y may not be nonpositively curved. So we must find an alternative to using the "visual boundary" of Y. This is done by constructing a certain family of smooth measures μ_s on Y itself, pushing them forward via $\widetilde{f}$, and convolving with Patterson-Sullivan measure on X. Maps $\widetilde{F}_s$ are then defined by taking the barycenters of these measures; it is actually these maps which are considered instead of F. This idea was first introduced in [2].

Two new features of F appear in higher rank. First, the non-strictness of convexity of the Busemann function (see §3) must be overcome to define F. Second, and more importantly, a theorem of Albuquerque shows that the support of each of the measures μ_x is codimension $\mathrm{rank}(X) - 1$ subset of ∂X called the *Furstenberg boundary* of X (see §2). This fact and its implications are crucial for later steps.

Step 2 (The Jacobian estimate): The heart of the paper (§4 and §5) is obtaining a universal bound on F, independent of f. For this, we first realize the Jacobian of F as the ratio of determinants of two matrix integrals. We then show that whenever there are small eigenvalues in the denominator there are a sufficient number of small eigenvalues in the numerator with which to cancel them. The key is to find these eigenvalues independently of the integrating measure (which depends on μ_s), therefore reducing the problem to a problem about semisimple Lie groups.

Step 3 (Finishing the proof): Once a universal bound on $|\operatorname{Jac}(F)|$ is found, a simple degree argument finishes the proof. In the case when M and N are not compact, the main difficulty is proving that F_s is proper. This is quite technical, and requires extending some of the ideas of [5] to the higher rank setting.

2. Patterson-Sullivan measures on symmetric spaces

In this section we briefly recall Albuquerque's theory [1] of Patterson-Sullivan measures in higher rank symmetric spaces. For background on nonpositively curved manifolds, symmetric spaces, visual boundaries, Busemann functions, etc., we refer the reader to [6] and [11].

2.1 Basic properties

Let X be a Riemannian symmetric space of noncompact type. Denote by ∂X the visual boundary of X; that is, the set of equivalence classes of geodesic rays in X, endowed with the cone topology. Hence $X \cup \partial X$ is a compactification of X which is homeomorphic to a closed ball.

The *volume entropy* $h(g)$ of a closed Riemannian n-manifold (M, g) is defined as

$$h(g) = \lim_{R\to\infty} \frac{1}{R} \log(\operatorname{Vol}(B(x, R)))$$

where $B(x,R)$ is the ball of radius R around a fixed point x in the universal cover X. The number $h(g)$ is independent of the choice of x, and equals the topological entropy of the geodesic flow on (M,g) when the curvature $K(g)$ satisfies $K(g) \leq 0$. Note that while the volume $\mathrm{Vol}(M,g)$ is not invariant under scaling the metric g, the *normalized entropy*

$$\mathrm{ent}(g) = h(g)^n \,\mathrm{Vol}(M,g)$$

is scale invariant.

Let Γ be a lattice in $\mathrm{Isom}(X)$, so that $h(g_0) < \infty$ where (M,g_0) is $\Gamma\backslash X$ with the locally symmetric metric.

Generalizing the construction of Patterson-Sullivan, Albuquerque constructs in [1] a family of *Patterson-Sullivan measures* on ∂X. This is a family of probability measures $\{\nu_x\}_{x\in X}$ on ∂X which provide a particularly natural embedding of X into the space of probability measures on ∂X.

Theorem 2.1 (Existence Theorem, [1])**.** *There exists a family $\{\nu_x\}$ of probability measures on ∂X satisfying the following properties:*

1. *Each ν_x has no atoms.*
2. *The family of measures $\{\nu_x\}$ is Γ-equivariant:*

$$\gamma_*\nu_x = \nu_{\gamma x} \textit{ for all } \gamma \in \Gamma.$$

3. *For all $x, y \in X$, the measure ν_y is absolutely continuous with respect to ν_x. In fact the Radon-Nikodym derivative is given explicitly by:*

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{h(g_0)B(x,y,\xi)} \tag{1}$$

where $B(x,y,\xi)$ is the Busemann function *on X. For points $x, y \in X$ and $\xi \in \partial X$, the function $B : X \times X \times \partial X \to \mathbf{R}$ is defined by*

$$B(x,y,\xi) = \lim_{t\to\infty} d_X(y, \gamma_\xi(t)) - t$$

where γ_ξ is the unique geodesic ray with $\gamma(0) = x$ and $\gamma(\infty) = \xi$.

The third property implies no two measures are the same as measures. Thus the assignment $x \mapsto \nu_x$ defines an injective map

$$\nu : X \to \mathcal{M}(\partial X)$$

where $\mathcal{M}(\partial X)$ is the space of probability measures on X. Such a mapping satisfying the above properties is called an $h(g_0)$-*conformal density.*

2.2 Symmetric spaces of noncompact type

Before we present Albuquerque's theorem we will need some necessary background about higher rank symmetric spaces.

By definition, the symmetric space X is G/K where G is a semisimple Lie group and K a maximal compact subgroup. Fix once and for all a basepoint $p \in X$. This choice uniquely determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G where $\mathfrak{k}$ is the Lie algebra of the isotropy subgroup $K = \mathrm{Stab}_G(p)$ of p in G and $\mathfrak{p}$ is orthogonal to $\mathfrak{k}$ with respect to the killing form $B(\cdot,\cdot)$ on $\mathfrak{g}$. Therefore, $\mathfrak{p}$ is also identified with the tangent space T_pX.

Let $\mathfrak{a}$ be, once and for all, a fixed maximal abelian subalgebra of $\mathfrak{g}$. It follows from the Cartan decomposition that $\mathfrak{a} \subset \mathfrak{p}$. The set $\exp(\mathfrak{a}) \cdot p$ will be a maximal flat (totally geodesically embedded Euclidean space of maximal dimension) in X. Recall, a vector $v \in TX$ is called a *regular vector* if it is tangent to a unique maximal flat. Otherwise it is a *singular vector*. A geodesic is called regular (resp. singular) if one (and hence all) of its tangent vectors are regular (singular). A point $\xi \in \partial X$ is regular (singular) if any (and hence all) of the geodesics in the corresponding equivalence class are regular (singular).

Let $\mathfrak{a}^*$ be the dual to $\mathfrak{a}$, then for each $\alpha \in \mathfrak{a}^*$ define

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} | \operatorname{ad}_A Y = \alpha(A)Y \text{for all } A \in \mathfrak{a}\}.$$

We call α a *root* if $\mathfrak{g}_\alpha \neq 0$. Therefore the roots form a finite set Λ.

If θ_p is the Cartan involution associated to the point p, which is Id on $\mathfrak{k}$ and $-\mathrm{Id}$ on $\mathfrak{p}$, then we may define a positive definite inner product ϕ_p on $\mathfrak{g}$ by $\phi_p(Y,Z) = -B(\theta_p Y, Z)$. With respect to ϕ_p, the folowing *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha$$

is orthogonal.

The following is proposition can be found in 2.7.3 of [11].

Proposition 2.2. *Some properties of the roots and root space decomposition are:*

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *if* $\alpha + \beta \in \Lambda$ *or is* 0 *otherwise.*

2. *If* $\alpha \in \Lambda$ *then* $-\alpha \in \Lambda$ *and* $\theta_p : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ *is an isomorphism.*

3. *If* α *is not an integer multiple of some other* $\lambda \in \Lambda$ *then the only possible multiples of* α *in* Λ *are* $\pm\alpha$ *and* $\pm 2\alpha$*.*

4. *We have* $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) + \mathfrak{a}$*.*

5. *If* $\alpha, \beta \in \Lambda$ *then* $\beta - 2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \in \Lambda$ *where* $\langle \cdot, \cdot \rangle$ *is the dual inner product to* ϕ_p *on* $\mathfrak{a}^*$*. Furthermore,* $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}$ *is always an integer and if* α *and* β *are not collinear then it is* ± 1*.*

We call a subset $\Delta \subset \Lambda$ a *base* for Λ if:

1. The elements of Δ form a basis (over $\mathbf{R}$) for $\mathfrak{a}^*$.

2. Every root in Λ can be written as a linear combination of of elements in Δ with coefficients being either all nonnegative integers or all nonpositive integers.

If we choose an regular element $A \in \mathfrak{a}$ then define the set of *positive roots* corresponding to A,

$$\Lambda_A^+ = \{\alpha \in \Lambda \mid \alpha(A) \geq 0\}$$

The subset $\Delta_A^+ \subset \Lambda_A^+$ consisting of elements which cannot be written as a sum of two elements in Λ_A^+ is a base for Λ. Sometimes Δ_A^+ is called a *fundamental system of positive roots*.

For $A \in \mathfrak{a}$ the associated (open) *Weyl chamber* $W(A)$ is the connected component of the set of regular vectors in $\mathfrak{a}$ which contains A. We also call the set $\exp W(A) \subset \exp(\mathfrak{a})$, as well as $\exp(W(A)) \cdot p \subset X$, a Weyl chamber which we again denote by $W(A)$ using the context to determine where exactly it lies.

The union of all the singular geodesics in the flat $\exp(\mathfrak{a}) \cdot p$ passing through p is a finite set of hyperplanes forming the boundaries of the Weyl chambers. This provides another description of the Weyl chamber $W(A)$ as

$$W(A) = \{Y \in \mathfrak{a} \mid \alpha(Y) > 0 \text{ for all } \alpha \in \Delta_A^+\}.$$

For each subset $I \subset \Delta_A^+$ the set $W_I(A) = \cap_{\alpha \in I}(\ker \alpha \cap \overline{W(A)}$ is called the *Weyl chamber face* corresponding to the set I, and we designate $W_\emptyset(A) = W(A)$. The subgroup of K which stabilizes the face $W_I(A)$ we denote by K_I.

2.3 The Furstenberg boundary

The *Furstenberg boundary* of a symmetric space X of noncompact type is abstractly defined to be G/P where P is a minimal parabolic subgroup of the connected component G of the identity in $\mathrm{Isom}(X)$.

The Furstenberg boundary can be identified with the orbit of G acting on any regular point $v(\infty) \in \partial X$, the endpoint of a geodesic tangent to a regular vector v. of a Weyl chamber in a fixed flat $\mathfrak{a}$. This follows from the fact that the action of any such P on ∂X fixes some regular point.

Because of this, for symmetric spaces of higher rank, behaviour on the visual boundary can often be aptly described by its restriction to the Furstenberg boundary. Here we will use only some very basic properties of this boundary. For more details on semisimple Lie groups and the Furstenberg boundary, see [23].

For a fixed regular vector $A \in \mathfrak{a}$ and associated set of positive roots Λ_A^+ the *barycenter* b of the Weyl chamber $W(A)$ is defined to be

$$b = \sum_{\alpha \in \Lambda_A^+} m_\alpha H_\alpha$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$ is the multiplicty of α and H_α is the dual vector (with respect to ϕ_p) of α. Set $b^+ = b/\|b\|$.

Define the set $\partial_F X \subset \partial X$ to be $\partial_F X = G \cdot b^+(\infty)$. Henceforth we will refer to the Furstenberg boundary as this specific realization. We point out that for any lattice Γ in $\mathrm{Isom}(X)$, the induced action on the boundary is transitive only on $\partial_F X$. That is, $\overline{\Gamma \cdot b^+(\infty)} = G \cdot b^+(\infty)$, even though for any interior point $x \in X$ we have $\overline{\Gamma \cdot x} = \partial X$.

2.4 Albuquerque's Theorem

Theorem 7.4 and Proposition 7.5 of [1] combine to give the following theorem, which will play a crucial role in our proof of Theorem 1.2.

Theorem 2.3 (Description of ν_x)**.** *Let (X, g_0) be a symmetric space of noncompact type, and let Γ be a lattice in* $\mathrm{Isom}(X)$*. Then:*

1. $h(g_0) = \|b\|$.
2. $b^+(\infty)$ *is a regular point, and hence $\partial_F X$ is a regular set.*
3. *For any $x \in X$, the support* $\mathrm{supp}(\nu_x)$ *of ν_x is equal to $\partial_F X$.*

4. *ν_x is the unique probability measure invariant under the action on $\partial_F X$ of the compact isotropy group* $\mathrm{Stab}_G(x)$ *at x. In particular, ν_p is the unique K-invariant probability measure on $\partial_F X$.*

Note that when X is has rank one, $\partial_F X = \partial X$. In general $\partial_F X$ has codimension $\mathrm{rank}(X) - 1$ in ∂X.

2.5 Limits of Patterson-Sullivan measures

We now describe the asymptotic behaviour of the ν_x as x tends to a point in ∂X.

For any point ξ of the visual boundary, let S_θ be the set of points $\xi \in \partial_F X$ such that there is a Weyl chamber W whose closure $\partial\overline{W}$ in ∂X contains both θ and ξ. Let K_θ be the subgroup of K which stabilizes S_θ. K_θ acts transitively on S_θ (see the proof below).

Theorem 2.4 (Support of ν_x)**.** *Given any sequence $\{x_i\}$ tending to $\theta \in \partial X$ in the cone topology, the measures ν_{x_i} converge in $\mathcal{M}(\partial_F X)$ to the unique K_θ-invariant probability measure ν_θ supported on S_θ.*

Proof. Let $x_i = g_i \cdot p$, for an appropriate sequence $g_i \in G$. Recall that $\nu_{x_i} = (g_i)_* \nu_p$. Then combining part (4) of Theorem 2.3 with Proposition 9.43 of [13] have that some subsequence of the ν_{x_i} converges to a K_θ-invariant measure ν_θ supported on S_θ.

Note that in [13], the notation I refers to a subset of a fundamental set of roots corresponding to the face of a Weyl chamber containing θ in its boundary. If $g_i \cdot p = k_i a_i \cdot p$ converges then both $k = \lim k_i$ and $a^I = \lim_i a_i^I$ exist (note the definition of a^I in [13]). Again in the notation of [13], K_θ is the conjugate subgroup $(ka^I)K^I(ka^I)^{-1}$ in K. Moreover, S_θ is the orbit $ka^I K^I \cdot b^+(\infty)$.

By Corollary 9.46 and Proposition 9.45 of [13] any other convergent subsequence of the ν_{x_i} produces the same measure in the limit, and therefore the sequence ν_{x_i} itself converges to ν_θ uniquely. q.e.d.

In the case when θ is a regular point, the above theorem implies that S_θ is a single point and the limit measure ν_θ is simply the Dirac probability measure at that point point in $\partial_F X$.

3. The barycenter of a measure

In this section we describe the natural map which is an essential ingredient in the method of Besson-Courtois-Gallot.

Let ϕ denote the lift to universal covers of f with basepoint $p \in Y$ (resp. $f(p) \in X$), i.e., $\phi = \widetilde{f} : Y \to X$. We will also denote the metric and Riemannian volume form on universal cover Y by g and dg respectively. Then for each $s > h(g)$ and $y \in Y$ consider the probability measure μ_y^s on Y in the Lebesgue class with density given by

$$\frac{d\mu_y^s}{dg}(z) = \frac{e^{-sd(y,z)}}{\int_Y e^{-sd(y,z)}dg}.$$

The μ_y^s are well-defined by the choice of s.

Consider the push-forward $\phi_*\mu_y^s$, which is a measure on X. Define σ_y^s to be the convolution of $\phi_*\mu_y^s$ with the Patterson-Sullivan measure ν_z for the symmetric metric.

In other words, for $U \subset \partial X$ a Borel set, define

$$\sigma_y^s(U) = \int_X \nu_z(U)d(\phi_*\mu_y^s)(z).$$

Since $\|\nu_z\| = 1$, we have

$$\|\sigma_y^s\| = \|\mu_y^s\| = 1.$$

Let $B(x,\theta) = B(\widetilde{f}(p), x, \theta)$ be the Busemann function on X with respect to the basepoint $\widetilde{f}(p)$ (which we will also denote by p). For $s > h(g)$ and $x \in X, y \in Y$ define a function

$$\mathcal{B}_{s,y}(x) = \int_{\partial X} B(x,\theta)d\sigma_y^s(\theta).$$

By Theorem 2.4, the support of ν_z, hence of σ_y^s, is all of $\partial_F X$, which in turn equals the G-orbit $G \cdot b^+(\infty)$. Hence

$$\mathcal{B}_{s,y}(x) = \int_{\partial_F X} B(x,\theta)d\sigma_y^s(\theta) = \int_{G\cdot b^+(\infty)} B(x,\theta)d\sigma_y^s(\theta).$$

Since X is nonpositively curved, the Busemann function B is (non-strictly) convex on X. Hence $\mathcal{B}_{s,y}$ is convex on X, being a convex integral of convex functions. While B is strictly convex only when X is negatively curved, we have the following.

Proposition 3.1 (Strict convexity of $\mathcal{B}$)**.** *For each fixed y and s, the function $x \mapsto \mathcal{B}_{y,s}(x)$ is strictly convex, and has a unique critical point in X which is its minimum.*

Proof. It suffices to show that given a geodesic segment $\gamma(t)$ between two points $\gamma(0), \gamma(1) \in X$, there exists some $\xi \in \partial_F X$ such that function $B(\gamma(t), \xi)$ is *strictly* convex in t, and hence on an open positive μ_y-measure set around ξ. We know it is convex by the comment preceding the statement of the proposition.

If $B(\gamma(t), \xi)$ is constant on some geodesic subsegment of γ for some ξ, then γ must lie in some flat $\mathcal{F}$ such that the geodesic between $\xi \in \partial\mathcal{F}$ and γ (which meets γ at a right angle) also lies in $\mathcal{F}$. On the other hand, $\xi \in \partial_F X$ is in the direction of the algebraic centroid in a Weyl chamber, and γ is perpendicular to this direction. By the properties of the roots, γ is a regular geodesic (i.e., γ is not contained in the boundary of a Weyl chamber). In particular, γ is contained in a unique flat $\mathcal{F}$. Furthermore, $\partial_F X \cap \partial\mathcal{F}$ is a finite set (an orbit of the Weyl group). As a result, for almost every $\xi \in \partial_F X$ $B(\gamma(t), \xi)$ is *strictly* convex in t.

For fixed $z \in X$, by the last property listed in Theorem 2.1, we see that

$$\int_{\partial_F X} B(x, \theta) d\nu_z(\theta)$$

tends to ∞ as x tends to any boundary point $\xi \in \partial X$. Then for fixed y and $s > h(g)$, $\mathcal{B}_{y,s}(x)$ increases to ∞ as x tends to any boundary point $\xi \in \partial X$. Hence it has a local minimum in X, which by strict convexity must be unique. q.e.d.

We call the unique critical point of $\mathcal{B}_{s,y}$ the *barycenter* of the measure σ_y^s, and define a map $\widetilde{F}_s : Y \to X$ by

$$\widetilde{F}_s(y) = \text{the unique critical point of } \mathcal{B}_{s,y}.$$

Since for any two points $p_1, p_2 \in X$

$$B(p_1, x, \theta) = B(p_2, x, \theta) + B(p_1, p_2, \theta)$$

we see that $\mathcal{B}_{s,y}$ only changes by an additive constant when we change the basepoint of B. Also, $\mathcal{B}_{s,y}$ only changes by a multiplicative constant when we change the basepoint in the definition of μ_y. Since neither change affects the critical point of $\mathcal{B}_{y,s}$, we see that $\widetilde{F}_s$ is independent of choice of basepoints.

The equivariance of $\widetilde{f}$ and of $\{\mu_y\}$ implies that $\widetilde{F}_s$ is also equivariant. Hence $\widetilde{F}_s$ descends to a map $F_s : N \to M$. It is easy to see that F_s is homotopic to f.

Proposition 3.2. *The map $\Psi_s\colon [0,1]\times N \to M$ defined by*

$$\Psi_s(t,y) = F_{s+\frac{t}{1-t}}(y)$$

is a homotopy between $\Psi_s(0,\cdot) = F_s$ and $\Psi_s(1,\cdot) = f$ for any $s > h(g)$.

Proof. From its definitions, $\widetilde{F}_s(y)$ is continuous in s and y. If (s_i, y_i) is a sequence converging to (s_0, y) for $s_0 \geq s$ and $y \in M$ then from the definition it is easy to verify that $\widetilde{F}_{s_i}(y_i)$ converges to $\widetilde{F}_{s_0}(y)$. If on the other hand $s_i \to \infty$, then observe that $\lim_{i\to\infty} \sigma_{y_i}^{s_i} = \nu_{\phi(y)}$. If follows that $\lim_{i\to\infty} \widetilde{F}_{s_i}(y_i) = \phi(y)$. This implies the proposition. q.e.d.

As in [2], we will see that F_s is C^1, and will estimate its Jacobian.

4. The Jacobian estimate

Let X be expressed as a product of its irreducible factors $X = X_1 \times \cdots \times X_k$, and let g_i denote the restricted symmetric metric on each factor X_i. As above, $h(g_i)$ denotes the volume entropy of (X_i, g_i). The main estimate of this paper is the following.

Theorem 4.1 (The Jacobian Estimate)**.** *For all $s > h(g)$ and all $y \in N$ we have*

$$|\operatorname{Jac} F_s(y)| \leq C\left(\frac{s}{h(g_1)h(g_2)\dots h(g_k)}\right)^n$$

for some constant C, depending only on $\dim M$.

Dependence of constants. Up to scaling of the metric, there are only a finite number of irreducible symmetric spaces of noncompact type in a given dimension. Therefore it is sufficient to show that C depends only on the individual symmetric spaces (X_i, g_i). Furthermore, when we apply Theorem 4.1, we will take the limit as $s \to h(g)$ so that the quantity $C\left(\frac{h(g)}{h(g_1)h(g_2)\dots h(g_k)}\right)^n$ is the constant appearing in Theorem 1.2. It is evident then that the right-hand side of inequality of Theorem 1.2 is scale invariant with respect to the metrics g and g_i.

We claim that the quantities $h(g)$ and $h(g_i)$ can be bounded by Ricci curvatures. The Bishop Volume Comparison Theorem ([7]) states that if the Ricci curvatures of (Y, g) are all greater than $(n-1)\kappa$ for some $\kappa \leq 0$ then for any $y \in Y$ and $r > 0$,

$$\operatorname{Vol} B(y, r) \leq V_\kappa(r)$$

where $V_\kappa(r)$ is the volume of the ball of radius r in the space form of constant curvature κ. In particular this implies that

$$h(g) \leq \lim_{r\to\infty} \frac{\log V_\kappa(r)}{r} = (n-1)\sqrt{-\kappa}.$$

Similarly, in the course of the proof of Theorem 4.1 we will see explicitly that

$$h(g_i) = \mathrm{Tr}\ \sqrt{-R_i(b^+,\cdot,b^+,\cdot)}$$

where R_i is the curvature tensor on (X_i, g_i). In particular

$$h(g_i) \geq \min\{1, -\mathrm{Ricci}(b^+, b^+)\}.$$

Therefore the constant C in Theorem 1.2 depends only on the Ricci curvatures of N and M.

We will prove Theorem 4.1 in several steps.

4.1 Finding the Jacobian

We obtain the differential of F_s by implicit differentiation:

$$0 = D_{x=F_s(y)}\mathcal{B}_{s,y}(x) = \int_{\partial_F X} dB_{(F_s(y),\theta)}(\cdot)d\sigma_y^s(\theta).$$

Hence as 2-forms

$$\begin{aligned}
0 &= D_y D_{x=F_s(y)}\mathcal{B}_{s,y}(x) \\
&= \int_{\partial_F X} DdB_{(F_s(y),\theta)}(D_y F_s(\cdot),\cdot)d\sigma_y^s(\theta) \\
&\quad - s\int_Y \int_{\partial_F X} dB_{(F_s(y),\theta)}(\cdot)\left\langle \nabla_y d(y,z),\cdot\right\rangle d\nu_{\phi(z)}(\theta)d\mu_y^s(z).
\end{aligned}$$

The distance function $d(y,z)$ is Lipschitz and C^1 off of the cut locus which has Lebesgue measure 0. It follows from the Implicit Function Theorem (see [3]) that F_s is C^1 for $s > h(g)$. By the chain rule,

$$\mathrm{Jac}\, F_s = s^n \frac{\det\left(\int_Y \int_{\partial_F X} dB_{(F_s(y),\theta)}(\cdot)\left\langle \nabla_y d(y,z),\cdot\right\rangle d\nu_{\phi(z)}(\theta)d\mu_y^s(z)\right)}{\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot)d\sigma_y^s(\theta)\right)}.$$

Applying Hölder's inequality to the numerator gives:

$$|\operatorname{Jac} F_s| \leq s^n \frac{\det\left(\int_{\partial_F X} dB^2_{(F_s(y),\theta)} d\sigma^s_y(\theta)\right)^{1/2} \det\left(\int_Y \langle \nabla_y d(y,z), \cdot\rangle^2 \, d\mu^s_y(z)\right)^{1/2}}{\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma^s_y(\theta)\right)}.$$

Using that Tr $\langle \nabla_y d(y,z), \cdot\rangle^2 = |\nabla_y d(y,z)|^2 = 1$, except possibly on a measure 0 set, we may estimate

$$\det\left(\int_Y \langle \nabla_y d(y,z), \cdot\rangle^2 \, d\mu^s_y(z)\right)^{1/2} \leq \left(\frac{1}{\sqrt{n}}\right)^n.$$

Therefore

$$|\operatorname{Jac} F_s| \leq \left(\frac{s}{\sqrt{n}}\right)^n \frac{\det\left(\int_{\partial_F X} dB^2_{(F_s(y),\theta)} d\sigma^s_y(\theta)\right)^{1/2}}{\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma^s_y(\theta)\right)}. \tag{2}$$

4.2 Reduction to the irreducible case

In this subsection we make, following [9], a reduction to the case when $X = \widetilde{M}$ is irreducible.

If $X = X_1 \times \cdots \times X_k$ is the irreducible expression for X as a product, the group $G = \operatorname{Isom}(X)$ can also be written as a product $G = G_1 \times G_2 \cdots \times G_k$, where each $G_i \neq \mathrm{SL}(2,\mathbf{R}), \mathrm{SL}(3,\mathbf{R})$ is a simple Lie group. Theorem 2.3 implies that for all $y \in Y$, the measure σ^s_y is supported on the G-orbit

$$G \cdot b^+(\infty) = \{(G_1 \times G_2 \cdots \times G_k) \cdot b^+(\infty)\}.$$

Hence

$$\partial_F X = G \cdot b^+(\infty) = \partial_F X_1 \times \cdot \times \partial_F X_k.$$

Since each X_i has rank one, $\partial_F X_i = \partial X_i$ so that

$$\partial_F X = \partial X_1 \times \cdots \times \partial X_k.$$

Let B_i denote the Busemann function for the rank one symmetric space X_i with metric g_i. Then for $\theta_i \in \partial X_i \subset \partial X$ and $x, y \in X_i$ we have $B(x,y,\theta_i) = B_i(x,y,\theta_i)$. Since the factors X_i are orthogonal in X

with respect to the metric g_0, the Busemann function of (X, g_0) with basepoint $p \in X$ at a point $\theta = (\theta_1, \ldots, \theta_k) \in \partial_F X$ is given by

$$B(x, \theta) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} B_i(x_i, \theta_i).$$

The Schur estimate for the determinant of symmetric semidefinite block matrices states,

$$\det \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \leq \det(A)\det(C).$$

Applying the dual form of this estimate to our symmetric tensors we have

$$\begin{aligned} &\det\left(\int_{\partial_F X} \left(\sum_{i=1}^{k} d(B_i)_{(\pi_i F_s(y), \pi_i\theta)}\right)^2 d\sigma_y^s(\theta)\right) \\ &\leq \prod_{i=1}^{k} \det\left(\int_{\partial_F X_i} \left(d(B_i)_{(\pi_i F_s(y), \theta_i)}\right)^2 d(\pi_i)_* \sigma_y^s(\theta_i)\right), \end{aligned}$$

where $\pi_i : X \to X_i$ and $\pi_i : \partial_F X \to \partial_F X_i$ are the canonical projections.

Since $DdB_{(F_s(y),\theta)} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} DdB_{i(\pi_i F_s(y), \pi_i\theta)}$, the denominator already splits as,

$$\begin{aligned} &\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma_y^s(\theta)\right) \\ &= \prod_{i=1}^{k} \det\left(\int_{\partial_F X_i} \left(Dd(B_i)_{(\pi_i F_s(y), \theta_i)}\right) d(\pi_i)_* \sigma_y^s(\theta_i)\right). \end{aligned}$$

Putting these together we obtain,

$$\begin{aligned} &|\operatorname{Jac} F_s(y)| \\ &\leq \left(\frac{s}{\sqrt{n}}\right)^n \prod_{i=1}^{k} \frac{\det\left(\int_{\partial_F X_i} \left(d(B_i)_{(\pi_i F_s(y),\theta_i)}\right)^2 d(\pi_i)_* \sigma_y^s(\theta_i)\right)^{1/2}}{\det\left(\int_{\partial_F X_i} \left(Dd(B_i)_{(\pi_i F_s(y),\theta_i)}\right) d(\pi_i)_* \sigma_y^s(\theta_i)\right)}. \end{aligned}$$

Therefore we only need to bound each term in the product seperately. It suffices then to prove that for an irreducible symmetric space $(X, g_0) \neq$

$\mathbf{H}^2, \mathrm{SL}(3,\mathbf{R})/\,\mathrm{SO}(3,\mathbf{R})$, and for any measure μ on $\partial_F X$, that

$$\frac{\det\left(\int_{\partial_F X} dB^2_{(F_s(y),\theta)} d\mu(\theta)\right)^{1/2}}{\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\mu(\theta)\right)} \leq \frac{C}{h(g_0)}.$$

We will continue to write σ^s_y instead of μ or $(\pi_i)_*\sigma^s_y$. The only property we use of σ^s_y from this point on is that it is fully supported on $\partial_F X$. Since $\mathrm{supp}((\pi_i)_*\sigma^s_y) = \pi_i(\mathrm{supp}(\sigma^s_y)) = \partial_F X_i$ there is no harm by this imprecision.

4.3 Simplifying the Jacobian

As stated above we need only now consider irreducible (X, g_0). For each point $x \in X$, we let $\mathcal{F}_x$ denote the canonical flat passing through x, i.e., $\mathcal{F}_x = \exp(\mathfrak{a}) \cdot x$. We denote the tangent space to $\mathcal{F}_x$ simply as $\mathcal{F}$ with the base point suppressed since it is naturally isomorphic to the Lie algebra $\exp(\mathfrak{a})$.

We wish to bound the quantity

$$\frac{\det\left(\int_{\partial_F X} dB^2_{(F_s(y),\theta)} d\sigma^s_y(\theta)\right)^{1/2}}{\det\left(\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma^s_y(\theta)\right)}.$$

Let $\mathcal{F}$ denote the tangent space to the flat $\mathcal{F}_{F_s(y)}$. Choose an orthonormal basis $\{e_i\}$ for the tangent space $T_{F_s(y)}X$ such that $e_1, \ldots,$ $e_{\mathrm{rank}(X)}$ is a basis for $\mathcal{F}$ with $e_1(\infty) = b^+(\infty)$. We may write the term

$$\int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma^s_y(\theta) \tag{3}$$

in matrix form as

$$\int_{\partial_F X} O_\theta \begin{pmatrix} 0 & 0 \\ 0 & D_\lambda \end{pmatrix} O^*_\theta \; d\sigma^s_y(\theta)$$

where O_θ is the orthogonal matrix in the e_i basis corresponding to the derivative of the unique isometry in $K = \mathrm{Stab}_G(F_s(y))$ which sends e_1 to $v_{(F_s(y),\theta)}$ (the vector in the tangent space of the point $F_s(y)$ in the direction $\theta \in \partial_F X$). In the above expression, the upper left zero matrix sub-block has dimensions $\operatorname{rank}(X) \times \operatorname{rank}(X)$, and D_λ has the form

$$D_\lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_{n-\operatorname{rank}(X)} \end{pmatrix}$$

where $\{\lambda_1, \dots, \lambda_{n-\operatorname{rank}(X)}\}$ is the set of nonzero eigenvalues of $DdB_{(F_s(y),\theta)}$. Since $DdB_{(x,\theta)}$ is G equivariant, its eigenvalues do not depend on x but only on which K-orbit in ∂X the point θ lies in. In particular, $DdB_{(x,\theta)}$ is flow invariant and hence the Ricatti equation shows that it is simply related to the curvature tensor by

$$DdB_{(x,\theta)} = \sqrt{-R(v_{(x,\theta)}, \cdot, v_{(x,\theta)}, \cdot)}.$$

On the other hand in a symmetric space $R(v, \cdot, v, \cdot) = -(\operatorname{ad}_v)^2|_{\mathfrak{p}}$. Therefore the eigenvalues of $DdB_{(F_s(y),\theta)}$ are those of $DdB_{(p,b^+(\infty))}$ which in turn are those of $\sqrt{\operatorname{ad}^2_{b^+}|_{\mathfrak{p}}}$. (Note that while ad_{b^+} does not preserve $\mathfrak{p}$, $(\operatorname{ad}_{b^+})^2|_{\mathfrak{p}}$ is a symmetric endomorphism of $\mathfrak{p}$.) Recall, $b^+ = b/\|b\|$ where $b = \sum_{\beta\in\Lambda^+_A} m_\beta H_\beta$ for any choice of $A \in \mathfrak{a}$ (the choice of A only determines the Weyl chamber containing b). Setting

$$\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

we have $\mathfrak{p}_\alpha = \{X - \theta_p X : X \in \mathfrak{g}_\alpha\}$.

By definition of $\mathfrak{g}_\alpha$, for each $\alpha \in \Lambda^+_A$ we may write

$$(\operatorname{ad}_{b^+})^2|_{\mathfrak{g}_\alpha} = \alpha(b^+)^2 \operatorname{Id} = \left(\frac{1}{\|b\|} \sum_{\beta\in\Lambda^+_A} \alpha(m_\beta H_\beta) \right)^2 \operatorname{Id}.$$

The same expression clearly holds for $(\operatorname{ad}_{b^+})^2|_{\mathfrak{g}_{-\alpha}}$. Therefore, for any $\alpha \in \Lambda$, $\sqrt{(\operatorname{ad}_{b^+})^2|_{\mathfrak{p}_\alpha}} = |\alpha(b^+)|$. For $\mathfrak{p}_0 = \mathfrak{a}$ the same formula holds with $\alpha = 0$. In particular, the ratio of the largest eigenvalue (denoted by $\lambda_{\max}$) among the λ_i's in D_λ to the smallest nonzero eigenvalue (denoted by $\lambda_{\min}$) only depends on X.

Furthermore, since $\alpha(b^+) > 0$ for all $\alpha \in \Lambda^+_A$ and $\dim \mathfrak{p}_\alpha = m_\alpha$, we have

$$\begin{aligned}
\operatorname{Tr} \sqrt{\operatorname{ad}_{b^+}^2 |_{\mathfrak{p}}} &= \sum_{\alpha \in \Lambda_A^+} m_\alpha \alpha(b^+) \\
&= \frac{1}{\|b\|} \sum_{\alpha, \beta \in \Lambda_A^+} m_\alpha m_\beta \alpha(H_\beta) \\
&= \frac{1}{\|b\|} \left\langle \sum_{\beta \in \Lambda_A^+} m_\beta H_\beta, \sum_{\alpha \in \Lambda_A^+} m_\alpha H_\alpha \right\rangle = \frac{\|b\|^2}{\|b\|} = h(g_0)
\end{aligned}$$

where the last equality follows from Theorem 2.3. As a result, there is a constant c only depending on X such that

$$\frac{h(g_0)}{c} \leq \lambda_i \leq c\, h(g_0) \tag{4}$$

for $i = 1, \ldots, (n - \operatorname{rank}(X))$. We now use the following.

Lemma 4.2. *The determinant of a sum of $n \times n$ positive semidefinite matrices is a nondecreasing homogeneous polynomial of degree n in the eigenvalues of each summand. Furthermore, if the sum is positive definite, then the determinant is strictly increasing in the eigenvalues of the summands.*

Proof. Let M be the sum of positive semidefinite matrices. Then there exist fixed orthogonal matrices O_l and real numbers $\lambda_{l,j}$ such that M may be written as

$$M = \sum_l O_l \begin{pmatrix} \lambda_{l,1} & 0 & \ldots & 0 \\ 0 & \lambda_{l,2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_{l,n} \end{pmatrix} O_l^*.$$

Then we have the differentiation formula (see, e.g., Prop. 2.8 of [8]):

$$\frac{d}{d\lambda_{l,j}} \det M = \operatorname{Tr} \left(\frac{d}{d\lambda_{l,j}} M \right) M^{\mathrm{adj}}$$

where M^{adj} is the adjunct matrix of M. Now,

$$\frac{d}{d\lambda_{l,j}} M = O_l E_{(j,j)} O_l^*$$

where $E_{(j,j)}$ is the elementary matrix with 1 in the (j,j) position and zeros elsewhere. Therefore, by cyclically permuting O_l in the trace above we find that $\frac{d}{d\lambda_{l,j}} \det M$ is the (j,j) the entry of $O_l^* M^{\mathrm{adj}} O_l$ which is nonnegative since M is positive semidefinite. Lastly, if M is positive definite then $O_l^* M^{\mathrm{adj}} O_l$ is also, which means that $\frac{d}{d\lambda_{l,j}} \det M$ is positive. The lemma follows. q.e.d.

Applying Lemma 4.2 to the Riemann sums for the integral (3) above, using the bound in Equation (4), and taking limits, gives

$$\begin{aligned}&\det \int_{\partial_F X} DdB_{(F_s(y),\theta)}(\cdot,\cdot) d\sigma_y^s(\theta)\\ &\geq \left(\frac{h(g_0)}{c}\right)^n \det \int_{\partial_F X} O_\theta \begin{pmatrix} 0 & 0 \\ 0 & I_{n-\mathrm{rank}(X)} \end{pmatrix} O_\theta^* \; d\sigma_y^s(\theta)\end{aligned}$$

where $I_{n-\mathrm{rank}(X)}$ is the identity matrix of dimension $n - \mathrm{rank}(X)$.

Next we observe that, relative to the orthonormal basis $\{e_1, \ldots, e_{\mathrm{rank}(X)}\}$ for $T_{F_s(y)}X$, the expression

$$\int_{\partial_F X} dB^2_{(F_s(y),\theta)} d\sigma_y^s(\theta)$$

may be written in the form

$$Q_1 = \int_{\partial_F X} O_\theta \begin{pmatrix} 1 & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 0_{(n-1)\times(n-1)} \end{pmatrix} O_\theta^* \; d\sigma_y^s(\theta)$$

where O_θ is the same matrix as above. Let

$$Q_2 = \int_{\partial_F X} O_\theta \begin{pmatrix} 0 & 0 \\ 0 & I_{n-\mathrm{rank}(X)} \end{pmatrix} O_\theta^* \; d\sigma_y^s(\theta).$$

We have just shown that, to prove Theorem 4.1, it suffices to prove that

$$\frac{\det Q_1}{(\det Q_2)^2} \leq C \tag{5}$$

for some constant C. The rest of this section will be devoted to proving this.

4.4 Eigenvalue matching

Here is the general idea of our proof of Theorem 4.1, which we have reduced to showing (5) above. Since the numerator is bounded above, we consider when the matrix Q_2 in the denominator has any eigenvalues smaller than a certain constant depending only on the dimension of X. When this occurs, Theorem 4.4 below will show that each such eigenvalue is matched by at least two smaller (up to a universal constant) eigenvalues of the matrix Q_1 in the numerator.

Let $\{v_i\}$ be an orthonormal eigenbasis for the symmetric matrix Q_2, and recall that $\{e_i\}$ is a basis for the tangent space $\mathcal{F}$ to the fixed, chosen flat. Note that the i-th eigenvalue of the matrix Q_2 may be written as

$$L_i = v_i^* Q_2 v_i = \int_{\partial_F X} \sum_{j=\mathrm{rank}(X)+1}^{n} \langle O_\theta . e_j, v_i \rangle^2 \, d\sigma_y^s(\theta).$$

We first argue that no L_i equals zero. Since $s > h(g)$ we have that the measures μ_y^s is a finite measure in the Lebesgue class (dg). Since the ν_x for $x \in X$ are positive on any open set (with respect to the cone topology) of $\partial_F X$, it follows that σ_y^s is as well. In particular, $\{O_\theta | \theta \in \mathrm{supp}(\sigma_y^s) = \partial_F X\}$ is isomorphic to the group K and therefore there is no nonzero subspace $V \subset T_{F_s(y)}X$ such that $O_\theta V \subset \mathcal{F}$ for all $\theta \in \partial_F X$. Hence none of the eigenvalues L_i are 0.

Let $\epsilon = 1/(\mathrm{rank}(X) + 1)$. Note that ϵ is a constant depending only on n, as there are only finitely many symmetric spaces of a given rank and given dimension. Suppose k of the eigenvalues are strictly less than ϵ. Since each $L_i \leq 1$, and since

$$\sum_i L_i = \mathrm{Tr}\ Q_2 = n - \mathrm{rank}(X)$$

it follows easily that $k \leq \mathrm{rank}(X)$. By rearranging the order we may assume that $L_i < \epsilon$ for $i = 1, \ldots, k$.

Let H be an inner product space over $\mathbf{R}$, and denote by $\mathrm{SO}(H)$ the special orthogonal group of H. Scale the bi-invariant metric on $\mathrm{SO}(H)$ so that $\mathrm{SO}(H)$ has diameter $\pi/2$. Define the *angle* between two subspaces $V, W \subset H$ as

$$\begin{aligned}&\angle(V, W)\\ &:= \inf \Big\{ d_{\mathrm{SO}(H)}(I, P) : P \in \mathrm{SO}(H) \text{ with } PV \subset W \text{ or } PW \subset V \Big\}.\end{aligned}$$

Let $\pi_V(W)$ represent the orthogonal projection of W onto V. Then it is routine to verify the following properties of the angle:

1. $\angle(V,W) \leq \frac{\pi}{2}$.
2. $\angle(V,W) = \angle(W^{\perp}, V^{\perp})$.
3. $\angle(V,W) = \angle(W,V)$.
4. If $V \subseteq U$ and $\dim U \leq \dim W$ then $\angle(V,W) \leq \angle(U,W)$, or if $V \subseteq U$ and $\dim V \geq \dim W$ then $\angle(V,W) \geq \angle(U,W)$.
5. If $\angle V, W = 0$ then $V \subseteq W$ or $W \subseteq V$.
6. If $U \subseteq W$ then $\angle(\pi_V(U), U) \leq \angle(\pi_V(U), W) \leq \angle(V,W)$.

For a 1-dimensional subspace V spanned by a vector v, our definition of angle agrees with the usual definition:

7. $V = \operatorname{span}\{v\} \Rightarrow \cos(\angle(V,W)) = \frac{\langle v, \pi_W(v)\rangle}{|v|\cdot|\pi_v(W)|}$.

Finally, $\angle$ satisfies the following form of the triangle inequality.

Lemma 4.3 (Triangle inequality for $\angle$)**.** *Let U, V, W be subspaces of a fixed inner product space H. Suppose that $\dim U = \dim W \leq \dim V$. Then*

$$\angle(V,W) \leq \angle(U,V) + \angle(U,W).$$

Proof. By definition of $\angle$ there exist $P_1, P_2, P_3 \in \mathrm{SO}(H)$ with:

- $P_1 W \subseteq V$ and $\angle(V,W) = d_{\mathrm{SO}(H)}(I, P_1)$.
- $P_2 U \subseteq V$ and $\angle(U,V) = d_{\mathrm{SO}(H)}(I, P_2)$.
- $P_3 U = W$ and $\angle(U,W) = d_{\mathrm{SO}(H)}(I, P_3)$.

Now $P_2 P_3^{-1} W \subseteq V$ so that

$$\begin{aligned} d(I, P_1) &\leq d(I, P_2 P_3^{-1}) \\ &= d(P_2, P_3) \\ &\leq d(I, P_2) + d(I, P_3) \end{aligned}$$

and we are done. q.e.d.

One of the main ingredients in the proof of Theorem 4.1 is the following.

Theorem 4.4 (Eigenvalue Matching Theorem)**.** *For any k-frame given by orthonormal vectors $v_1, \ldots, v_k$ of T_xX with $k \leq \mathrm{rank}(X)$ there is an orthonormal $2k$-frame given by vectors $v_1', v_1'' \ldots, v_k', v_k''$, each perpendicular to $\mathrm{span}\{v_1, \ldots, v_k\}$, such that for $i = 1, \ldots, k$ and all $h \in K$, there is a constant C, depending only on $\dim X$, such that*

$$\angle(hv_i', \mathcal{F}^\perp) \leq C\angle(hv_i, \mathcal{F})$$

and

$$\angle(hv_i'', \mathcal{F}^\perp) \leq C\angle(hv_i, \mathcal{F})$$

where hv represents the linear (derivative) action of K on $v \in T_xX$.

We will prove Theorem 4.4 in Section 5; its proof is independent of the rest of the paper.

4.5 Proof of the Jacobian estimate

Assuming Theorem 4.4 for the moment, we now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. From Equation (2) and the reduction in §4.3 we see that it is sufficient to show that

$$\frac{\det Q_1}{(\det Q_2)^2} \leq C$$

for some constant C depending only on n.

As before let $L_1, \ldots, L_k$ be the $k \leq \mathrm{rank}(X)$ eigenvalues of Q_2 which are strictly less than $\epsilon = 1/(\mathrm{rank}(X) + 1)$. If no such eigenvalues exist, then there is a lower bound on Q_2 depending only on $\mathrm{rank}(X)$. As there is an upper bound on Q_1, we are done (see the discussion on dependency of constants above). So we assume $k \geq 1$.

Let $v_1, \ldots, v_k$ be an orthonormal set of associated eigenvectors. Recall that $\{e_i\}$ denotes the chosen orthonormal basis for the $T_{F_s(y)}X$ such that $e_1, \ldots, e_{\mathrm{rank}(X)}$ spans the tangent space $\mathcal{F}$ to the fixed maximal flat.

For any vector $v \in T_{F_s(y)}X$ let

$$r(v) = \sum_{j=\mathrm{rank}(X)+1}^{n} \langle e_j, v\rangle^2$$

so that

$$L_i = \int_{\partial_F X} r(O_\theta^* v_i)\ d\sigma_y^s(\theta).$$

Since $e_1, \dots, e_{\mathrm{rank}(X)}$ form an orthonormal basis for $\mathcal{F}$, for any unit vector v we have

$$\begin{aligned}
\cos(\angle(v,\mathcal{F})) &= \langle v, \pi_{\mathcal{F}}(v)\rangle / |\pi_{\mathcal{F}}(v)| \\
&= \left\langle v, \sum \langle v, e_j\rangle e_j \right\rangle \Big/ \left(\sum \langle v, e_j\rangle^2\right)^{1/2} \\
&= \left(\sum \langle v, e_j\rangle^2\right)^{1/2}
\end{aligned}$$

so that

$$\cos(\angle(v,\mathcal{F}))^2 = \sum_{j=1}^{\mathrm{rank}(X)} \langle v, e_j\rangle^2 .$$

Hence

$$\begin{aligned}
r(v) &= 1 - \sum_{j=1}^{\mathrm{rank}(X)} \cos^2(\angle v, e_j) \\
&= 1 - \cos^2(\angle v, \mathcal{F}) \\
&= \sin^2(\angle v, \mathcal{F}).
\end{aligned}$$

Similarly

$$\langle v, e_1\rangle^2 \leq \sum_{j=1}^{\mathrm{rank}(X)} \langle v, e_j\rangle^2 = \sin^2(\angle v, \mathcal{F}^{\perp}).$$

For each $i = 1, \dots, k$, let v_i' and v_i'' be the pair of vectors corresponding to v_i produced by the Eigenvalue Matching Theorem (Theorem 4.4). That theorem together with the concavity of $\sin^2\theta$ for $0 \leq \theta \leq \pi/2$ gives, for all $\theta \in \partial_F X$ and for each $w_i = v_i'$ or v_i'', that

$$\sin^2(\angle O_\theta^* w_i, \mathcal{F}^{\perp}) \leq \sin^2(C\angle O_\theta^* v_i, \mathcal{F}) \leq C^2 \sin^2(\angle O_\theta^* v_i, \mathcal{F})$$

where $C > 1$ is the constant in the Eigenvalue Matching Theorem.

Furthermore, Q_1 is the integral (against a probability measure) of matrices with all eigenvalues less than 1 so no eigenvalue of Q_1 is greater than one. Hence we may estimate

$$\begin{aligned}
\det Q_1 &\leq \prod_{i=1}^{k} (v_i'.Q_1.v_i')(v_i''.Q_1.v_i'') \\
&= \prod_{i=1}^{k} \left(\int_{\partial_F X} \left\langle e_1, O_\theta^*.v_i' \right\rangle^2 \, d\sigma_y^s(\theta) \right) \\
&\qquad \cdot \left(\int_{\partial_F X} \left\langle e_1, O_\theta^*.v_i'' \right\rangle^2 \, d\sigma_y^s(\theta) \right) \\
&\leq \prod_{i=1}^{k} \left(\int_{\partial_F X} \sin^2(\angle O_\theta^*.v_i', \mathcal{F}^\perp) \, d\sigma_y^s(\theta) \right) \\
&\qquad \cdot \left(\int_{\partial_F X} \sin^2(\angle O_\theta^*.v_i'', \mathcal{F}^\perp) \, d\sigma_y^s(\theta) \right) \\
&\leq \prod_{i=1}^{k} \left(\int_{\partial_F X} C^2 \sin^2(\angle O_\theta^*.v_i, \mathcal{F}) \, d\sigma_y^s(\theta) \right) \\
&\qquad \cdot \left(\int_{\partial_F X} C^2 \sin^2(\angle O_\theta^*.v_i, \mathcal{F}) \, d\sigma_y^s(\theta) \right) \\
&= C^{2k} \prod_{i=1}^{k} L_i^2 \\
&= C^{2k} \det Q_2^2 \prod_{i=k+1}^{n} L_i^{-2} \\
&\leq C^{2k} \det Q_2^2 (\operatorname{rank}(X)+1)^{2(n-k)}.
\end{aligned}$$

The last inequality follows from the definition of k, whereby $L_i \geq \frac{1}{\operatorname{rank}(X)+1}$ for each $i > k$.

The constant C in Theorem 4.1 may be taken to be the product (over factors X_j of X with dimension n_j),

$$\frac{1}{\sqrt{n}^n} \prod_j C_j^{\operatorname{rank}(X)} c_j^{n_j} (\operatorname{rank}(X_j)+1)^{(n_j)}$$

where $C_j \geq 1$ is the constant C from Theorem 4.4, c_j is the constant c in Equation (4) and k_j is the constant k above. This combined constant depends only on $n = \dim X$. q.e.d.

4.6 A cautionary example

In the general method of [2] as well as here, one is solving a minimization problem without regard to the measure. However, at least in the $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$ case, to get a bound on the Jacobian of F_s one must use further properties of the measure, as indicated by the example we now give.

If for a single flat $\mathcal{F}_x$ and a sequence of $y_i \in \mathcal{F}_x$, the measures $\sigma^s_{y_i}$ tend to the sum of Dirac measures $\frac{1}{2}\delta_{b^+(\infty)} + \frac{1}{2}\delta_{wb^+(\infty)}$ where w is in the Weyl group for $\mathcal{F}_x$, then we claim that $\operatorname{Jac} F_s(y_i) \to \infty$. First note that the sum

$$dB^2_{(F_s(y_i),b^+(\infty))} + dB^2_{(F_s(y_i),wb^+(\infty))}$$

has only a 3-dimensional kernel, while

$$DdB_{(F_s(y_i),b^+(\infty))} + DdB_{(F_s(y_i),wb^+(\infty))}$$

has a 2-dimensional kernel. Furthermore

$$Q_1 = \int_{\partial_F X} dB^2_{(F_s(y_i),\theta)} d\sigma^s_{y_i} \quad \text{and} \quad Q_2 = \int_{\partial_F X} DdB_{(F_s(y_i),\theta)} d\sigma^s_{y_i}$$

degenerate in the same way, so that $\det(Q_1)/\det(Q_2)^2$ is unbounded. This can be easily verified explicitly in the case of a sum of five Dirac measures for which both integrals are nonsingular degenerating to the sum of the two Dirac measures given above.

A similar problem occurs when there are $\mathbf{H}^2$ factors. These and other examples are worked out in full detail in Section 6 of [10].

5. Proof of the Eigenvalue Matching Theorem

In order to prove Theorem 4.4 we will need a series of lemmas.

5.1 Dimension inequalities

For any $x \in X$ and any subspace $V \subseteq T_x X$, denote by K_V the elements of K which stabilize V (i.e., leave V invariant). For $V \subset \mathcal{F}$, if $\mathrm{Fix}_K(V)$ is the subgroup of K which fixes V pointwise then $K_V = U \cdot \mathrm{Fix}_K(V)$ where U is the subgroup stabilizing V of the (discrete) Weyl group which stabilizes $\mathcal{F}$ (see [11]).

The following lemma is a basic algebraic ingredient in the proof of Theorem 4.4.

Lemma 5.1 (Dimension inequality, I). *With the above notations,*

$$\dim\left(\operatorname{span}\{K_V\cdot\mathcal{F}\}^{\perp}\right)\geq 2\dim(V).$$

Proof. First we show that $K_V\cdot\mathcal{F}$ is itself a subspace hence equal to its span.

Recognize that $K_V\cdot\mathcal{F}$ is the union of all tangent spaces to flats which contain V. Pick a basis $v_1,\ldots,v_l$ of V note that $K_V\cdot\mathcal{F}=\cap_{i=1}^{l}\mathcal{F}(v_i)$ where $\mathcal{F}(v_i)$ is the union of all the tangent spaces to flats containing v_i using the notation of [11]. Proposition 2.11.4 of [11] states that $\mathcal{F}(v_i)=\mathbf{R}^r\times X_i$ for some symmetric space of noncompact type and $r\leq\operatorname{rank}(X)$. In particular it is a manifold and the tangent space to it corresponds to $K_{v_i}\cdot\mathcal{F}$, which is a vector space. Then $K_V\cdot\mathcal{F}$ is a vector space.

Let $K_{\mathcal{F}}$ be the stabilizer of $\mathcal{F}$ in K. Then $K_{\mathcal{F}}\subset W\cdot K_V$ where W denotes the Weyl group (a finite group). Hence $\dim K_{\mathcal{F}}=\dim(K_{\mathcal{F}}\cap K_V)$. Hence

$$\dim K_V\cdot\mathcal{F}=\dim K_V+\dim\mathcal{F}-\dim K_{\mathcal{F}}.$$

Since $X=K\cdot\mathcal{F}$ we obtain

$$\dim M=\dim K+\dim\mathcal{F}-\dim K_{\mathcal{F}}.$$

Putting this together we obtain,

$$(\dim\operatorname{span}\{K_V\cdot F\})^{\perp}=\dim M-\dim K_V\cdot\mathcal{F}=\dim K-\dim K_V.$$

But Lemma 5.2 below gives that this final term is $\geq 2\dim V$, as desired. q.e.d.

The following lemma was used in the proof of Lemma 5.1. Recall that, at this point, we are assuming that the symmetric space X is irreducible and has $\operatorname{rank}(X)\geq 2$.

Lemma 5.2 (Dimension inequality, II). *Assume that $X\neq\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. Then for any subspace $V\subset\mathcal{F}$, we have*

$$\dim K\geq 2\dim V+\dim K_V.$$

This lemma is the only place where $X\neq\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$ is used.

Proof. For a root $\alpha\in\Lambda$ in $\mathcal{F}$, define $\mathfrak{k}_\alpha=(\mathrm{Id}+\theta_p)\mathfrak{g}_\alpha$, where θ_p is the Cartan involution at $p=F_s(y)$. Then by Proposition 2.14.2 of [11] we have that $\mathfrak{k}_\alpha=\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}\cap\mathfrak{k}$, $\mathfrak{k}_\alpha=\mathfrak{k}_{-\alpha}$, and $\dim\mathfrak{k}_\alpha\geq 1$.

Note that from the definition of $\mathfrak{g}_\alpha$ it follows immediately that

$$\mathfrak{k}_\alpha = \{Y \in \mathfrak{k} | [X, Y] = 0 \text{ for all } X \in \ker \alpha\}.$$

Note that in G the normalizer mod centralizer is finite for any flat subspace. Therefore for any $V \subset \mathcal{F}$ we may write the Lie algebra $\mathfrak{k}_V$ of K_V as,

$$\mathfrak{k}_V = \{Y \in \mathfrak{k} | [X, Y] = 0 \text{ for all } X \in V\}.$$

It then follows from the previous statements that,

$$\mathfrak{k}_V = \mathfrak{k}_0 + \sum_{\substack{\alpha \in \Lambda \\ V \subset \ker \alpha}} \mathfrak{k}_\alpha.$$

Consequently, we may assume that V in the statement of the lemma is *maximally singular*: V may be written as the intersection of the kernels of the greatest number of roots among all subspaces of dimension $\dim V$. Otherwise $\dim K_V = \dim \mathfrak{k}_V$ is strictly smaller than it would be if V were maximally singular.

Recall that we have the invariant inner product ϕ_p on $\mathfrak{a}$ and hence on $\mathcal{F}$. Let Λ denote the collection of roots. For $\alpha \in \Lambda$, let $H_\alpha \in \mathcal{F}$ denote the dual root vector (with respect to ϕ_p) corresponding to α. For any subset $V \subset \mathcal{F}$ we define the function

$$\mathrm{card}_R(V) := \frac{1}{2} \mathrm{card}\{\alpha \in \Lambda | H_\alpha \in V\}.$$

Since root vectors lying in a subspace always come in opposing pairs, card_R is a positive integer.

Let α be any root. Note that if a subspace $V \subset \ker \alpha$, then H_α lies in $V^\perp$. Therefore the statement of the lemma reduces to showing that

$$\dim \mathfrak{k}_0 + \sum_{\alpha \in \Lambda} \dim \mathfrak{k}_\alpha \geq 2 \dim V + \dim \mathfrak{k}_0 + \sum_{\substack{\alpha \in \Lambda \\ V \subset \ker \alpha}} \dim \mathfrak{k}_\alpha$$

or more simply,

$$\sum_{H_\alpha \in \mathfrak{F} \setminus V^\perp} \dim \mathfrak{k}_\alpha \geq 2 \dim V.$$

Swapping $V^\perp$ for V and vice versa, and using $\dim \mathfrak{k}_\alpha \geq 1$ for each α, it is sufficient to prove that

$$\mathrm{card}_R(\mathcal{F} \setminus V) \geq 2(\mathrm{rank}(X) - \dim V). \tag{6}$$

Since we are assuming that G is simple, we could check this condition by using a classification of root vectors in the simple algebras such as in [22]. However, because this would be tedious we will instead give a synthetic proof.

For each $i = 0, \ldots, \mathrm{rank}(X)$, we say that $W_i \subset \mathcal{F}$ is a *maximally rooted subspace of dimension* i if

$$\mathrm{card}_R(W_i) = \max\{\mathrm{card}_R(V) : V \subset \mathcal{F} \text{ with } \dim V = i\}.$$

In other words, W_i is maximally rooted if $W_i^{\perp}$ is maximally singular. We claim that if $0 = W_0, W_1, \ldots, W_{\mathrm{rank}(X)} = \mathcal{F}$ are any maximally rooted subspaces of $\mathcal{F}$ with $\dim W_i = i$, then for $0 < i \leq \mathrm{rank}(X)$,

$$\mathrm{card}_R(W_i) \geq i + \mathrm{card}_R(W_{i-1}). \tag{7}$$

This is true for $i = 1$ since W_1 is one dimensional it contains a root vector pair and the trivial subspace W_0 contains none. By induction, assume the claim holds for all maximally rooted subspace W_i of dimension $i < j$. In particular, for such a space W_{j-1} and for any subspace $Z \subset W_{j-1}$ of codimension one, $\mathrm{card}_R(Z) \leq \mathrm{card}_R(W_{j-2})$ so

$$\mathrm{card}_R(W_{j-1} \setminus Z) = \mathrm{card}_R(W_{j-1}) - \mathrm{card}_R(Z) \geq j - 1.$$

We claim that there exists a root vector H_α which is not in W_{j-1} or its perpendicular $W_{j-1}^{\perp}$ (with respect to ϕ_p). If not, then every root vector either lies in W_{j-1} or $W_{j-1}^{\perp}$ which implies the root system is reducible (e.g., Corollary 27.5 of [18]), and hence G is reducible, contrary to assumption.

Therefore, $H_\alpha^{\perp} \cap W_{j-1}$ is a codimension one subspace of W_{j-1} and by inductive hypothesis there are at least $j-1$ distinct pairs of root vectors $\pm H_{\alpha_1}, \ldots, \pm H_{\alpha_{j-1}}$ in $W_{j-1} \setminus (H_\alpha^{\perp} \cap W_{j-1})$. For each of these we have $\phi_p(H_\alpha, H_{\alpha_l}) \neq 0$. By the standard calculus of roots (e.g., Proposition 2.9.3 of [11]) this implies that for each $1 \leq l \leq j-1$ either $\pm(H_\alpha + H_{\alpha_l})$ or $\pm(H_\alpha - H_{\alpha_l})$ is a pair of root vectors lying in $W_{j-1} \oplus \langle H_\alpha \rangle$ which does not lie in W_{j-1}. Including H_α, these form at least j pairs of root vectors which are contained in $W_{j-1} \oplus \langle H_\alpha \rangle \setminus W_{j-1}$. Therefore $\mathrm{card}_R(W_{j-1} \oplus \langle H_\alpha \rangle) \geq \mathrm{card}_R(W_{j-1}) + j$. Since by definition of W_j, $\mathrm{card}_R(W_j) \geq \mathrm{card}_R(W_{j-1} \oplus \langle H_\alpha \rangle)$, the claim follows.

Recursively applying Equation (7) shows that for $0 \leq i < j \leq \mathrm{rank}(X)$,

$$\mathrm{card}_R(W_j) - \mathrm{card}_R(W_i) \geq \sum_{k=i}^{j} k = \frac{j(j+1)}{2} - \frac{i(i+1)}{2}.$$

Now to prove the inequality (6), as noted before we may assume V of dimension q is maximally rooted, since then $V^{\perp}$ is maximally singular. Since $\mathcal{F}$ is a maximally rooted space, the above expression reads

$$\begin{aligned}\operatorname{card}_R(\mathcal{F}\setminus V) &= \operatorname{card}_R(\mathcal{F}) - \operatorname{card}_R(V)\\ &= \frac{\operatorname{rank}(X)(\operatorname{rank}(X)+1)}{2} - \frac{q(q+1)}{2}.\end{aligned}$$

This is readily seen to be greater that $2(\operatorname{rank}(X)-q)$ unless $\operatorname{rank}(X)$ $= 2$ and $q = 0$ $(V = \mathcal{F})$. However, every irreducible Lie algebras of rank two other than $\mathfrak{sl}(3,\mathbf{R})$ has at least four pairs of roots (see [17], p. 44, Figure 1), and hence the inequality (6) is satisfied in all of the required cases. q.e.d.

5.2 Angle inequalities

Lemma 5.3 (Angle inequality, I)**.** *For any subspace $V \subseteq \mathcal{F}$ there is a subspace $V' \subset V^{\perp}$ with $\dim V' \geq 2\dim V$ and a constant C depending only on the symmetric space X such that for all $k \in K$,*

$$\angle(kV', \mathcal{F}^{\perp}) \leq C\angle(kV, \mathcal{F})$$

where kV represents the linear (derivative) action of K on $V \subset T_xX$.

Proof. For any subspace $V \subset \mathcal{F}$, let $U_1, U_2, \ldots, U_{l(V)}$ be the maximally singular subspaces of dimension $\dim V$ which have minimal angle with V. Define $S_V = U_1 \oplus \ldots \oplus U_{l(V)} \subset \mathcal{F}$. If $G(r,\mathcal{F})$ denotes the Grassmann variety of subspaces in $\mathcal{F}$ with dimension r, then the set of $V \in G(r,\mathcal{F})$ for which $l(V)$ is constant has codimension $l(V)-1$ in $G(r,\mathcal{F})$.

For any subspace $V \subset \mathcal{F}$ we define a subspace $V' \subset \mathcal{F}^{\perp}$ by

$$V' = (\operatorname{span}\{K_{S_V}\cdot\mathcal{F}\})^{\perp}$$

where K_{S_V} is the subgroup of K which stabilizes S_V. By Proposition 5.1, V' has dimension at least $2\dim V$ since we always have $K_{S_V} \subset K_U$ for some $U \subset \mathcal{F}$ with $\dim U = \dim V$.

If no such constant C as in the lemma exists then there is a sequence $k_i \in K$ and $V_i \subset \mathcal{F}$ with $\dim V_i = r$ such that

$$\frac{\angle(k_i V_i, \mathcal{F})}{\angle(k_i V_i', \mathcal{F}^{\perp})} \to 0. \tag{8}$$

Now since S_V and hence V' varies upper semicontinuously in V (thinking of the map $V \to V'$ as a self-map of $G(r,\mathcal{F})$), it follows from the continuity of the $\angle$ function that

$$\frac{\angle(k\,V,\mathcal{F})}{\angle(k\,V',\mathcal{F}^{\perp})}$$

is lower semicontinuous in V.

However since both K and $G(r,\mathcal{F})$ are compact, for some subsequence of the k_iV_i, the k_i converge to $k_0 \in K$ and the V_i converge to a fixed subspace $V_0 \subset \mathcal{F}$. Furthermore, k_0V_0 lies in $\mathcal{F}$ since $\angle(k_0V_0,\mathcal{F})$ must be 0. It follows that $k_0 \in W \cdot K_{V_0}$ where W is the Weyl group stabilizing $\mathcal{F}$.

By construction, $K_{V_0} \subset K_{V_0'}$ and for any $w \in W$,

$$\angle(wV_0',\mathcal{F}^{\perp}) = \angle(V_0', w^{-1}\mathcal{F}^{\perp}) = \angle(V_0',\mathcal{F}^{\perp}).$$

Therefore, we also have $\angle(k_0V_0',\mathcal{F}^{\perp}) = 0$. Continuity of $\angle$ along with the fact that $W \subset K$ acts isometrically implies that it is sufficient to show that for any fixed subspace $V \subset \mathcal{F}$ the quantity

$$\liminf_{k\to K_V} \frac{\angle(kV,\mathcal{F})}{\angle(kV',\mathcal{F}^{\perp})}$$

is bounded away from 0. Note that since this quantity is lower semicontinuous in V, and since $G(r,\mathcal{F})$ is compact, it is unnecessary to show that the bound is independent of V.

First we handle the denominator. Using the bi-invariance of the metric on $\mathrm{SO}(n)$, the properties of the angle function, and the fact that for all $k_0 \in K_{S_V}$ we have $k_0\,k\,k^{-1}\mathcal{F} \subset K_{S_V}\mathcal{F}$, it follows that

$$\begin{aligned}
&d_{\mathrm{SO}(n)}(k, K_{S_V})\\
&= d_{\mathrm{SO}(n)}(k^{-1}, K_{S_V})\\
&= d_{\mathrm{SO}(n)}(K_{S_V}\cdot k, Id)\\
&\geq \inf\{d_{\mathrm{SO}(n)}(\mathrm{Id}, P) : P \in \mathrm{SO}(n) \text{ with } Pk^{-1}\mathcal{F} \subset K_{S_V}\mathcal{F}\}\\
&= \angle(\mathrm{span}\{K_{S_V}\mathcal{F}\}, k^{-1}\mathcal{F})\\
&= \angle(kK_{S_V}\mathcal{F},\mathcal{F})\\
&= \angle\left((kK_{S_V}\mathcal{F})^{\perp},\mathcal{F}^{\perp}\right)\\
&= \angle(kV',\mathcal{F}^{\perp}).
\end{aligned}$$

So it remains to show that for any sequence $k_i \to k^V \in K_V$ in any fixed neighborhood U of K_V, that $\angle(k_i V, \mathcal{F}) \geq C d_{\mathrm{SO}(n)}(k_i, K_{S_V})$. Furthermore, since $\angle(k_i V, \mathcal{F}) = \angle(k_i (k_i^V)^{-1} V, \mathcal{F})$ for any $k_i^V \in K_V$, we may assume that $k_i \to \mathrm{Id}$.

By Theorem 2.10.1 of [22], in a sufficiently small neighborhood of Id we may uniquely write k_i as $k_i = \exp(\mathrm{k}_i^{\perp}) \exp(\mathrm{k}_i^S)$ where $\mathrm{k}_i^S \in \mathfrak{k}_{S_{V_0}}$ and $\mathrm{k}_i^{\perp} \in \mathfrak{k}_{S_{V_0}}^{\perp}$. Furthermore $\mathrm{k}_i^S \to 0$ and $\mathrm{k}_i^{\perp} \to 0$.

Bi-invariance of the metric on $\mathrm{SO}(n)$ implies that for $|\mathrm{k}_i^{\perp}| < \frac{\pi}{2}$,

$$d_{\mathrm{SO}(n)}(k_i, K_{S_V}) = d_{\mathrm{SO}(n)}(\exp(\mathrm{k}_i^{\perp}), K_{S_V}) = |\mathrm{k}_i^{\perp}|.$$

Now K_V is the only subgroup of K which both leaves V in $\mathcal{F}$ and also intersects all sufficiently small neighborhoods of the identity. Therefore, in order to show that $\angle(k_i V, \mathcal{F}) \geq C|\mathrm{k}_i^{\perp}|$, we need only show that

$$d_{\mathrm{SO}(n)}(k_i, K_V)/|k_i^{\perp}| \not\to 0.$$

Well, the Cambell-Baker-Hausdorff formula implies that

$$\exp(\mathrm{k}_i^{\perp}) \exp(\mathrm{k}_i^S) = \exp\left(\mathrm{k}_i^{\perp} + \mathrm{k}_i^S + O(|\mathrm{k}_i^{\perp}| \cdot |\mathrm{k}_i^S|)\right).$$

Since the definition of S_V implies that $\mathfrak{k}_{S_V} \supset \mathfrak{k}_V$ and $\mathrm{k}_i^{\perp}$ is perpendicular to $\mathfrak{k}_{S_V}$, we have

$$d_{\mathrm{SO}(n)}(k_i, K_V) \geq |\mathrm{k}_i^{\perp}| + O(|\mathrm{k}_i^{\perp}| \cdot |\mathrm{k}_i^S|).$$

Since we had $|\mathrm{k}_i^S| \to 0$ this finishes the lemma. q.e.d.

Lemma 5.4 (Angle inequality, II). *For any subspace V of $T_x X$ with $\dim V \leq \mathrm{rank}(X)$, there is a subspace $V' \perp V$ with $\dim V' \geq 2 \dim V$, and a constant C depending only on n, such that*

$$\angle(kV', \mathcal{F}^{\perp}) \leq C \angle(kV, \mathcal{F}) \quad \textit{for all } k \in K.$$

Proof. The first step of the proof is to reduce to the case when V is a subspace of $\mathcal{F}$, so that Lemma 5.3 may be applied.

We first observe that the lemma is true if and only if it is true with V replaced by $k_0 V$ for any fixed $k_0 \in K$. Since K is compact we may therefore choose V among all $kV, k \in K$ so that $\angle(V, \mathcal{F}) \leq \angle(kV, \mathcal{F})$ for all $k \in K$.

With this assumption, consider the projection $W = \pi_F(V)$ of V onto $\mathcal{F}$. By Lemma 5.3, we obtain a subspace W' such that

$$\angle(kW', \mathcal{F}^\perp) \leq C\angle(kW, \mathcal{F})$$

for all $k \in K$. Then we let V' be the projection of W' onto $V^\perp$. By the properties of the angle function (see 4.4), it follows that

$$\begin{aligned} \angle(kV', \mathcal{F}^\perp) &\leq \angle(kW', \mathcal{F}^\perp) + \angle(kV', kW') && \text{by Lemma 4.3} \\ &\leq C\angle(kW', \mathcal{F}^\perp) + \angle(V', W') && \\ &\leq C\angle(kW, \mathcal{F}) + \angle(V', W') && \text{since } (W^\perp)^\perp \supseteq W \\ &\leq C\angle(kW, \mathcal{F}) + \angle V, \mathcal{F} && \text{for same reason} \\ &= C\angle(kW, \mathcal{F}) + \angle(V, \mathcal{F}) && \text{since } W = \pi_F(V). \end{aligned}$$

Thus it suffices to bound $\angle(kW, \mathcal{F})$ by a constant times $\angle(kV, \mathcal{F})$. But

$$\begin{aligned} \angle(kW, \mathcal{F}) &\leq \angle(kV, \mathcal{F}) + \angle(kV, kW) && \text{by Lemma 4.3} \\ &= \angle(kV, \mathcal{F}) + \angle(V, W) && \\ &= \angle(kV, \mathcal{F}) + \angle(V, \mathcal{F}) && \text{as } W = \pi_F(V) \\ &\leq \angle(kV, \mathcal{F}) + \angle(kV, \mathcal{F}) && \text{by minimality} \\ &= 2\angle(kV, \mathcal{F}) && \end{aligned}$$

and we are done. q.e.d.

5.3 Finishing the proof of the Eigenvalue Matching Theorem

Armed with the lemmas of the previous two subsections, we now prove Theorem 4.4.

We begin by noting that the construction of V' from V above respects subspace inclusion. I.e. if $U \subset V$ then $U' \subset V'$. This follows from the definition of V' and the fact that for two singular subspaces

U_1 and U_2 with $U_1 \subset U_2$, we have $K_{U_1} \cdot (W \cap K_{U_2}) \supset K_{U_2}$, where W is the Weyl group.

Now we simply proceed by induction on the number of vectors k. For $k = 1$ we set $V = v_1$ the statement of the proposition follows from Lemma 5.4. Order the vectors by increasing angle with $\mathcal{F}$. Assume the proposition for $k-1$ vectors, then set $V_k = \operatorname{span}\{v_1, \ldots, v_k\}$. By Lemma 5.4 we have an orthogonal subspace of twice the dimension of V_k, namely V_k', which we may write by the preceeding paragraph as $V_k' = V_{k-1}' \oplus W'$ where W' is two dimensional. The same lemma also guarantees that $\angle W', \mathcal{F}^\perp \leq C\angle v_k, \mathcal{F}$, since $\angle v_k, \mathcal{F} = \angle V_k, \mathcal{F}$.

This completes the proof of Theorem 4.4.

6. Finishing the proof of the Degree Theorem

We will break the proof of Theorem 1.2 into the compact and non-compact cases.

6.1 The compact case

Suppose M and N are compact. Since for $s > h(g)$, F_s is a C^1 map, using Proposition 3.2 and elementary integration theory yields,

$$
\begin{aligned}
|\deg(f)| \operatorname{Vol}(M) &= |\deg(f)| \int_M dg_0 \\
&= \left| \int_N f^* dg_0 \right| \\
&= \left| \int_N F_s^* dg_0 \right| \\
&\leq \int_N |\operatorname{Jac} F_s| \, dg \\
&\leq C \left(\frac{s}{h(g_0)} \right)^n \operatorname{Vol}(N).
\end{aligned} \tag{9}
$$

For the last inequality we have used the principal estimate from Theorem 4.1. Rearranging terms gives us the inequality in Theorem 1.2 since C depends only on the dimension and $\left(\frac{s}{h(g_0)}\right)^n$ depends only on n and the smallest Ricci curvatures of M and N.

6.2 The noncompact case

We now consider the case when N (and/or M) has finite volume but is not compact. In this setting, it is not known whether the limit in the definition of $h(g)$ always exists. For this reason we will define the quantity $h(g)$ to be

$$h(g) = \inf \left\{ s \geq 0 \,\middle|\, \exists C > 0 \text{ such that } \forall y \in Y,\ \int_Y e^{-sd(y,z)} dg(z) < C \right\}.$$

In fact this agrees with the previous definition for $h(g)$ when N is compact. In the case of the symmetric space (M, g_0) this definition of $h(g_0)$ agrees with the previous definition for compact manifolds.

For the finite volume case, the main difficulty is that, in order for the proof given above to work, we need to know that F_s is proper (and thus surjective since $\deg(F_s) = \deg(f) \neq 0$). For this, we will need to prove higher rank analogs of some lemmas used in [5] for the rank one case. For the basics of degree theory for proper maps between noncompact spaces, see [12]. We will need to assume that the geometry of N is bounded in the sense that its Ricci curvatures are bounded from above and that the injectivity radius of its universal cover Y is bounded from below. These are the specific assumptions implied in the third remark after the theorem.

We will show that F_s is proper by essentially showing that the barycenter of σ_y^s lies nearby a convex set containing large mass for this measure. This convex set is in turn far away from $\phi(p)$ whenever x is far from $p \in Y$. We achieve this by first estimating the concentration of the mass of σ_y^s in certain cones which will be our convex sets. One difficulty that arises in the higher rank is that these cones must have a certain angle when restricted to a flat. Another difficulty is that the ends of M can have large angle at infinity. In fact our methods breakdown unless we control the asymptotic expansion of f down the ends (see Remarks).

First, we localize the barycenter of the measure σ_y^s. Let $v_{(x,\theta)}$ be the unit vector in S_xX pointing to $\theta \in \partial X$.

Lemma 6.1. *Let $K \subset X$ and $y \in Y$ be such that $(\phi_* \mu_y^s)(K) > C$ for some constant $1 > C > \frac{1}{2}$. Suppose that for all $x \in X$ there exists $v \in S_xX$ such that for all $z \in K$:*

$$\int_{\partial_F X} \left\langle v_{(x,\theta)}, v \right\rangle d\nu_z(\theta) \geq \frac{1}{C} - 1.$$

Then

$$x \neq \widetilde{F}_s(y).$$

Proof. If $\widetilde{F}_s(y) = x$ then $\nabla_x \mathcal{B}_{s,y}(x) = 0$. However, $\nabla_x \mathcal{B}_{s,y}(x)$ may be expressed as

$$\int_X \int_{\partial_F X} v_{(x,\theta)} d\nu_z(\theta) d\phi_* \mu_y^s(z)$$

where $v_{(x,\theta)}$ is the unit vector in $S_x X$ pointing to $\theta \in \partial_F X$. Then we have

$$\begin{aligned}
\|D_x \mathcal{B}_{s,y}\| &= \left\| \int_X \int_{\partial_F X} v_{(x,\theta)} d\nu_z(\theta) d\phi_* \mu_y^s(z) \right\| \\
&\geq \left\| \int_K \int_{\partial_F X} v_{(x,\theta)} d\nu_z(\theta) d\phi_* \mu_y^s(z) \right\| \\
&\quad - \left\| \int_{X-K} \int_{\partial_F X} v_{(x,\theta)} d\nu_z(\theta) d\phi_* \mu_y^s(z) \right\| \\
&\geq \int_K \int_{\partial_F X} \left\langle v_{(x,\theta)}, v \right\rangle d\nu_z(\theta) d\phi_* \mu_y^s(z) - \phi_* \mu_y^s(X-K) \\
&\geq \phi_* \mu_y^s(K) \left(\frac{1}{C} - 1 \right) - 1 + \phi_* \mu_y^s(K) \\
&> C \left(\frac{1}{C} - 1 \right) - 1 + C = 0.
\end{aligned}$$

The strictness of the inequality finishes the proof. q.e.d.

For $v \in SX$ and $\alpha > 0$ consider the convex cone,

$$E_{(v,\alpha)} = \exp_{\pi(v)} \left\{ w \in T_{\pi(v)} X \mid \angle_{\pi(v)}(v(\infty), w(\infty)) \leq \alpha \right\},$$

where $\pi : TX \to X$ is the tangent bundle projection.

Denote by $\partial E_{(v,\alpha)} \subset \partial X$ its boundary at infinity.

Lemma 6.2. *There exists $T_0 > 0$ and $\alpha_0 > 0$ such that for all $t \geq T_0$, all $x \in X$, all $v \in S_x X$ and all $z \in E_{(g^t v, \alpha_0)}$,*

$$\int_{\partial_F X} \left\langle v_{(x,\theta)}, v \right\rangle d\nu_z(\theta) \geq \frac{\sqrt{2}}{3}.$$

Proof. Since the isometry group of the symmetric space X is transitive on X and for any isometry ψ, $d\psi(E_{(v,\alpha)}) = E_{(d\psi(v),\alpha)}$, it is sufficient to prove the lemma for a fixed x and all $v \in S_x X$.

For now choose $\alpha_0 < \pi/4$. Take a monotone sequence $t_i \to \infty$, and any choice $z_i \in E_{(g^{t_i}v,\alpha)}$ for each t_i. It follows that some subsequence of the z_i, which we again denote by $\{z_i\}$, must tend to some point $\theta \in \partial E_{(v,\alpha)}$.

Let ν_θ be the weak limit of the measures ν_{z_i}. From Theorem 2.4, ν_θ is a probability measure supported on a set S_θ satisfying

$$\angle_x(\theta, \xi) \leq \frac{\pi}{4} \quad \forall \xi \in S_\theta.$$

Therefore we have,

$$\int_{S_\theta} \left\langle v_{(x,\xi)}, v_{(x,\theta)} \right\rangle d\nu_\theta(\xi) \geq \frac{\sqrt{2}}{2}. \tag{10}$$

Now whenever $\theta \in \partial E_{(v,\alpha)}$ then $v = v_{(x,\theta)} + \epsilon v'$ for some unit vector v' and $\epsilon \leq \sin(\alpha)$. Using either case above we may write

$$\int_{\partial_F X} \left\langle v_{(x,\xi)}, v \right\rangle d\nu_\theta(\xi) \geq \int_{\partial_F X} \left\langle v_{(x,\xi)}, v_{(x,\theta)} \right\rangle d\nu_\theta(\xi) - \sin(\alpha).$$

So choosing α small enough we can guarantee that:

1. Any two Weyl chambers intersecting $E_{(g^t v,\alpha)}$ for all $t > 0$ in the same flat must share a common face of dimension $\mathrm{rank}(M) - 1$.

2. For any $\theta \in \partial E_{(v,\alpha)}$,

$$\int_{\partial_F X} \left\langle v_{(x,\xi)}, v \right\rangle d\nu_\theta(\xi) \geq \frac{\sqrt{2}}{2.5}.$$

Let

$$E_{(v(\infty),\alpha)} = \cap_{t>0} \partial E_{(g^t v,\alpha)}.$$

By the first property used in the choice of α above, for any two points $\theta_1, \theta_2 \in E_{(v(\infty),\alpha)}$, either θ_1 and θ_2 are in the boundary of the same Weyl chamber, or else there is another point θ' in the intersection of the boundaries at infinity of the closures of the respective Weyl chambers.

By maximality there is some $\theta_0 \in E_{(v(\infty),\alpha)}$ intersecting the boundary at infinity of the closure of every Weyl chamber which intersects $E_{(g^t v,\alpha)}$ for all $t > 0$. Hence, for every $\theta \in E_{(v(\infty),\alpha)}$, the support of the limit measure ν_θ satisfies $S_\theta \subset S_{\theta_0}$. (While θ_0 is not necessarily unique, the support S_{θ_0} of the corresponding limit measure ν_{θ_0} is.)

As t increases, for any $z \in E_{(g^t v,\alpha)}$, the measures ν_z uniformly become increasingly concentrated on S_{θ_0}. Then applying the estimate (10) to $\theta = \theta_0$, we may choose T_0 sufficiently large so that for all $z \in E_{(g^t v,\alpha)}$ with $t > T_0$,

$$\int_{\partial_F X} \left\langle v_{(x,\xi)}, v \right\rangle d\nu_z(\xi) \geq \frac{\sqrt{2}}{3}.$$

q.e.d.

Proposition 6.3. *F_s is proper.*

Proof. By way of contradiction, let $y_i \in Y$ be an unbounded sequence such that $\{\widetilde{F}_s(y_i)\}$ lies in a compact set K. We may pass to an unbounded subsequence of $\{y_i\}$, which we again denote as $\{y_i\}$, such that the sequence $\phi(y_i)$ converges within a fundamental domain for $\pi_1(M)$ in X to a point $\theta_0 \in \partial X$. Since K is compact, the set

$$A = \bigcap_{x \in K} E_{(g^{T_0} v_{(x,\theta_0)},\alpha_0)}$$

contains an open neighborhood of θ_0 and $d_X(A, K) \geq T_0$. Notice that A is itself a cone, being the intersection of cones on a nonempty subset of ∂X.

We now show that A contains the image $\phi(B(y_i, R_i))$ of increasingly large balls ($R_i \to \infty$). However, we observe from the fact that A is a cone on an open neighborhood of θ_0 in ∂X that A contains balls $B(\phi(y_i), r_i)$ with $r_i \to \infty$. By assumption f, and hence ϕ, is coarsely Lipschitz:

$$d_X(\phi x, \phi y) \leq K d_Y(x, y) + C$$

for some constants $C > 0$ and $K \geq 1$. Therefore $\phi^{-1}(B(\phi(y_i), r_i)) \supset B(y_i, R_i)$ where $KR_i + C > r_i$. In particular $R_i \to \infty$.

Hence, there exists an unbounded sequence R_i such that $B(y_i, R_i) \subset \phi^{-1}(A)$. Furthermore, since the Ricci curvature is assumed to be bounded from above and the injectivity radius from below, we have that $\mathrm{Vol}(B(y_i, \mathrm{injrad}))$ is greater than some constant independent of y_i and hence $\int_Y e^{-sd(y_i,z)} dg(z) > Q$ for some constant $Q > 0$. By choice of s there is a constant C_s depending only on s such that $\int_Y e^{-sd(y,z)} dg(z) < C_s$ for all $y \in Y$.

In polar coordinates we may write,

$$\begin{aligned}\int_Y e^{-sd(y,z)}dg(z) &= \int_0^\infty e^{-st}\,\mathrm{Vol}(S(y,t))dt \\ &= \int_0^\infty e^{-st}\frac{d}{dt}\,\mathrm{Vol}(B(y,t))dt \\ &= -\int_0^\infty \frac{d}{dt}\left(e^{-st}\right)\ \mathrm{Vol}(B(y,t))dt \\ &= s\int_0^\infty e^{-st}\,\mathrm{Vol}(B(y,t))dt.\end{aligned}$$

Using this we may estimate, using any $\delta < s - h(g)$,

$$\begin{aligned}\mu^s_{y_i}(\phi^{-1}(A)) &> \mu^s_{y_i}(B(y_i,R_i)) \\ &= 1 - \frac{\int_{R_i}^\infty e^{-st}\,\mathrm{Vol}(B(y_i,t))dt}{\int_0^\infty e^{-st}\,\mathrm{Vol}(B(y_i,t))dt} \\ &\geq 1 - \frac{e^{-\delta R_i}\int_{R_i}^\infty e^{-(s-\delta)t}\,\mathrm{Vol}(B(y_i,t))dt}{\int_0^\infty e^{-st}\,\mathrm{Vol}(B(y_i,t))dt} \\ &\geq 1 - e^{-\delta R_i}\frac{C_{s-\delta}}{Q}.\end{aligned}$$

Therefore for all sufficiently large i,

$$\mu^s_{y_i}(\phi^{-1}(A)) > \frac{3}{3+\sqrt{2}}.$$

The constant $\frac{3}{3+\sqrt{2}}$ is the constant C from Lemma 6.1 such that $\frac{1}{C} - 1 = \frac{\sqrt{2}}{3}$.

Set $v_i = g^{T_0+1}v_{(\widetilde{F}_s(y_i),\theta_0)}$. Recalling that $A \subset E_{(v_i,\alpha_0)}$ for all i, we have that for sufficiently large i,

$$\phi_*\mu^s_{y_i}(E_{(v_i,\alpha_0)}) > \frac{3}{3+\sqrt{2}}$$

but $d_X(\widetilde{F}_s(y_i), E_{(v_i,\alpha_0)}) > T_0$, contradicting the conclusion of Lemma 6.1 in light of Lemma 6.2. q.e.d.

Remarks.

1. In the proof of the above proposition, we used that injrad is bounded from below and Ricci curvature is bounded from above only to show that the volume of balls of any fixed radius are bounded from below.

2. Ideas from coarse topology can be used to remove the coarse Lipschitz assumption on f in the case that the ends of M have angle at infinity bounded away from $\pi/2$. However, M may have ends containing pieces of flats with wide angle (consider the product of two rank one manifolds each with multiple cusps, or for a classification of higher rank locally symmetric ends see [16]). For products of such surfaces it is possible, by expanding a family of infinite cones, to construct a proper map $f : M \to M$ such that for a radial sequence $y_i \to \xi \in \partial X$, ϕ maps the bulk of the mass of $\mu^s_{y_i}$ into a set (almost) symmetrically arranged about a point $p \in \widetilde{X}$ thus keeping $\widetilde{F}_s(y_i)$ bounded. This explains the need for a condition on f akin to the coarse Lipschitz hypothesis.

The last proposition actually shows that $d(F_{s_i}(x), f(x))$ is bounded for $s_i \geq s > h(g)$. In particular, the homotopy in Proposition 3.2 is proper. The inequality in Theorem 1.2 now follows as in the compact case, with $\deg(f)$ and $\deg(F_s)$ suitably interpreted.

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